

A temperature-dependent phase segregation problem of the Allen-Cahn type

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*Dedicated to Professor Nobuyuki Kenmochi
on the occasion of his 65th birthday*

Abstract

In this paper we prove a local-in-time existence theorem for an initial-boundary value problem related to a model of temperature-dependent phase segregation that generalizes the standard Allen-Cahn’s model. The problem is ruled by a system of two differential equations, one partial the other ordinary, interpreted as balances, respectively, of microforces and of microenergy, complemented by a transcendental condition on the three unknowns, that are: the order parameter entering the standard A-C equation, the chemical potential, and the absolute temperature. The results obtained in our recent paper [3] dealing with the isothermal case serve as a starting point for our existence proof, which relies on a fixed-point argument involving the Tychonoff-Schauder theorem.

Key words: Allen-Cahn equation; integrodifferential system; temperature variable; local existence.

AMS (MOS) Subject Classification: 74A15, 35K55, 35A01

1 Introduction

This paper is a sequel and a generalization of our article [3], where we studied a nonlinear evolution system of the Allen-Cahn (A-C) type, intended to provide a mathematical description of the phenomenology of phase segregation by atom rearrangement in crystalline materials, in the absence of diffusion. Our present generalization consists in taking thermal effects into account. To help understanding the underlying physics, temperature-independent mathematical models of A-C type and their proposed generalization are briefly discussed in the next section, where we recapitulate Gurtin's derivation of the standard A-C equation as well as the derivation of the nonstandard A-C system analyzed in [3], and where we sketch the main traits of our present temperature-dependent model. In Section 3, we formulate carefully the corresponding mathematical problem and we state a local-in-time existence result, that we prove in our last section.

2 Phase segregation models of A-C type

2.1 The temperature-independent model of [3]

In [3], we consider a nonlinear evolution system consisting of the partial differential equation:

$$\kappa \partial_t \rho - \Delta \rho + f'(\rho) = \mu \quad (2.1)$$

and of the ordinary differential equation:

$$\partial_t(-\mu^2 \rho) = \mu (\kappa (\partial_t \rho)^2 + \bar{\sigma}) , \quad (2.2)$$

complemented with the homogeneous Neumann boundary condition:

$$\partial_n \rho = 0 \quad \text{on the body's boundary} \quad (2.3)$$

(here ∂_n denotes the outward normal derivative) and with the initial conditions:

$$\rho|_{t=0} = \rho_0 \quad \text{bounded away from 0,} \quad \mu|_{t=0} = \mu_0 \geq 0. \quad (2.4)$$

The parabolic PDE (2.1) and the first-order-in-time ODE (2.2) are interpreted as balances, respectively, of microforces and of microenergy. They are to be solved for the order-parameter field $\rho = \rho(x, t) \in [0, 1]$, interpreted as the scaled volumetric density of one of two coexisting phases, and for the chemical potential field μ . Moreover, $\kappa > 0$ is a mobility coefficient, f is a double-well potential confined in $(0, 1)$ and singular at endpoints, and $\bar{\sigma} = \bar{\sigma}(x, t)$ denotes a given source term. The microentropy field $\eta = -\mu^2 \rho$ cannot exceed the level 0 from below, so that the corresponding prescribed initial field

$$\eta|_{t=0} = \eta_0 , \quad (2.5)$$

with $\eta_0 = -\mu_0^2 \rho_0$, is nonpositive-valued. Taking $\mu \equiv 0$ in (2.1) yields the standard Allen-Cahn equation, which is intended to describe evolutionary processes in a two-phase material body, *phase segregation* included.

2.2 Gurtin's derivation of the A-C equation

The derivation of the A-C equation proposed by Gurtin [6] (see also [5] and [7] for similar derivations and for discussions of related models) is based on a *balance of contact and distance microforces*:

$$\operatorname{div} \boldsymbol{\xi} + \pi + \gamma = 0 \quad (2.6)$$

and on a dissipation inequality restricting the free-energy growth:

$$\partial_t \psi \leq w, \quad w := -\pi \partial_t \rho + \boldsymbol{\xi} \cdot \nabla(\partial_t \rho), \quad (2.7)$$

where the distance microforce is split in an internal part π and an external part γ , the contact microforce is specified by the microscopic stress vector $\boldsymbol{\xi}$, and w is the (distance plus contact) internal microworking. Requesting the Coleman-Noll (C-N) compatibility [1] of the constitutive choices:

$$\pi = \widehat{\pi}(\rho, \nabla \rho, \partial_t \rho), \quad \boldsymbol{\xi} = \widehat{\boldsymbol{\xi}}(\rho, \nabla \rho, \partial_t \rho), \quad \text{and} \quad \psi = \widehat{\psi}(\rho, \nabla \rho) = f(\rho) + \frac{1}{2} |\nabla \rho|^2, \quad (2.8)$$

with the dissipation inequality (2.7) yields:

$$\widehat{\pi}(\rho, \nabla \rho, \partial_t \rho) = -f'(\rho) - \widehat{\kappa}(\rho, \nabla \rho, \partial_t \rho) \partial_t \rho \quad \text{and} \quad \widehat{\boldsymbol{\xi}}(\rho, \nabla \rho, \partial_t \rho) = \nabla \rho.$$

Under the further assumptions that $\widehat{\kappa}(\rho, \nabla \rho, \partial_t \rho) = \kappa$, a positive constant, and that $\gamma \equiv 0$, the microforce balance (2.6) takes the form of the standard Allen-Cahn equation

$$\kappa \partial_t \rho - \Delta \rho + f'(\rho) = 0. \quad (2.9)$$

2.3 The derivation of the A-C system of [3]

In [3], on adopting an approach put forward by one of us in [8], we deal with a modified version of Gurtin's derivation, in which the dissipation inequality (2.7) is dropped and the microforce balance (2.6) is coupled both with the *microenergy balance*

$$\partial_t \varepsilon = e + w, \quad e := -\operatorname{div} \bar{\mathbf{h}} + \bar{\sigma}, \quad (2.10)$$

and the *microentropy imbalance*

$$\partial_t \eta \geq -\operatorname{div} \mathbf{h} + \sigma, \quad \mathbf{h} := \mu \bar{\mathbf{h}}, \quad \sigma := \mu \bar{\sigma} \quad (2.11)$$

(here $\bar{\sigma}$ is the external source of energy per unit volume). With a view toward modeling phase-segregation, we postulate that the microentropy inflow (\mathbf{h}, σ) be proportional to the microenergy inflow $(\bar{\mathbf{h}}, \bar{\sigma})$ through the chemical potential μ , a positive field; consistently, we define the free energy to be:

$$\psi := \varepsilon - \mu^{-1} \eta, \quad (2.12)$$

with the chemical potential playing the same role as the coldness ϑ^{-1} in the deduction of the heat equation. Combination of (2.10)–(2.12) leads to the inequality:

$$\partial_t \psi \leq -\eta \partial_t (\mu^{-1}) + \mu^{-1} \bar{\mathbf{h}} \cdot \nabla \mu - \pi \partial_t \rho + \boldsymbol{\xi} \cdot \nabla(\partial_t \rho), \quad (2.13)$$

which replaces (2.7) as a restriction on constitutive choices. We assume that, in addition to the independent variables ρ , $\nabla\rho$ and $\partial_t\rho$, the constitutive mappings delivering π, ξ, η , and $\bar{\mathbf{h}}$ depend also on μ ; moreover, we choose:

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2. \quad (2.14)$$

To impose the C-N compatibility of these assumptions with (2.13) makes sense, because we have at our disposal two independent controls γ and $\bar{\sigma}$ to guarantee the free linear continuation in time of any given process $t \mapsto (\rho, \mu)(t)$ at any fixed space point. We find:

$$\begin{aligned} \widehat{\pi}(\rho, \nabla\rho, \partial_t\rho, \mu) &= \mu - f'(\rho) - \widehat{\kappa}(\rho, \nabla\rho, \partial_t\rho)\partial_t\rho, & \widehat{\xi}(\rho, \nabla\rho, \partial_t\rho, \mu) &= \nabla\rho, \\ \widehat{\eta}(\rho, \nabla\rho, \partial_t\rho, \mu) &= -\mu^2\rho, & \widehat{\mathbf{h}}(\rho, \nabla\rho, \partial_t\rho, \mu) &\equiv \mathbf{0}; \end{aligned} \quad (2.15)$$

the last of these findings implies that the microenergy balance – in general, a PDE – becomes an ODE, a crucial mathematical simplification that we exploit in the following just as we did in [3]. Finally, under the additional assumptions that the mobility is a positive constant and the external distance microforce is null, the microforce balance (2.6) and the energy balance (2.10) become, respectively, (2.1) and (2.2).

2.4 Accounting for thermal effects

As is well-known (see e.g. [8]), the classic heat equation can be arrived at by coupling the energy balance

$$\partial_t\varepsilon = -\operatorname{div}\bar{\mathbf{h}} \quad (2.16)$$

and the entropy imbalance

$$\partial_t\eta \geq -\operatorname{div}\mathbf{h}, \quad \mathbf{h} = \vartheta^{-1}\bar{\mathbf{h}}, \quad (2.17)$$

with the following constitutive prescriptions:

$$\psi = \varepsilon - \vartheta\eta, \quad \psi = \bar{\psi}(\vartheta) = -c_v\vartheta(\ln\vartheta - 1), \quad (2.18)$$

with the absolute temperature field ϑ positive-valued and the specific heat c_v a positive number. To account for thermal effects on the phenomenology of phase segregation by atomic rearrangement, we compare the formats (2.16)–(2.18) and (2.10)–(2.14). A way to match them, in the light of the relationships of temperature and chemical potential to entropy provided by statistical mechanics, is to assume that: (i) the microenergy balance keeps the form (2.10); (ii) the energy/entropy fluxes and the free energy have the mutually consistent forms

$$\mathbf{h} = (\vartheta^{-1}\mu)\bar{\mathbf{h}}, \quad \psi = \varepsilon - (\vartheta\mu^{-1})\eta, \quad \psi = \widehat{\psi}(\rho, \nabla\rho, \mu, \vartheta). \quad (2.19)$$

The second assumption is the main element of novelty of this note. With that measure, the dissipation inequality that replaces for (2.13) is:

$$\partial_t\psi \leq -\eta\partial_t(\vartheta\mu^{-1}) + (\vartheta\mu^{-1})\bar{\mathbf{h}} \cdot \nabla(\vartheta^{-1}\mu) - \pi\partial_t\rho + \xi \cdot \nabla(\partial_t\rho), \quad (2.20)$$

where, in addition to (2.19)₃ for the free energy, the distance force, the microscopic stress, the entropy, and the microenergy influx, are assumed to depend on the list of variables $\Lambda = \{\rho, \nabla \rho, \mu, \vartheta; \partial_t \rho, \nabla \vartheta\}$.¹

When C-N compatibility of the present constitutive prescriptions with (2.20) is inspected, a delicate modeling issue emerges: this time, we cannot count on as many controls as needed to guarantee the free local continuation in time of any given process $t \mapsto (\rho, \mu, \vartheta)(t)$ at any fixed space point. We do get the counterparts of the first, second, and fourth of (2.15), namely,

$$\widehat{\pi} = -\partial_\rho \psi - \kappa \partial_t \rho, \quad \widehat{\xi} = \partial_{\nabla \rho} \psi = \nabla \rho, \quad \widehat{\mathbf{h}} \equiv \mathbf{0}; \quad (2.21)$$

and we are left with the residual inequality:

$$(\partial_\mu \psi - \vartheta \mu^{-2} \eta) \partial_t \mu + (\partial_\vartheta \psi + \mu^{-1} \eta) \partial_t \vartheta \leq 0. \quad (2.22)$$

Now, if it were possible to choose both $\partial_t \mu$ and $\partial_t \vartheta$ arbitrarily, then (2.22) would yield the double equality:

$$\eta = \vartheta^{-1} \mu^2 \partial_\mu \psi = -\mu \partial_\vartheta \psi. \quad (2.23)$$

This observation motivates our decision to complement the microforce and energy balances with another field equation, namely, the *thermodynamic consistency condition*:

$$\mu \partial_\mu \psi + \vartheta \partial_\vartheta \psi = 0. \quad (2.24)$$

With this, (2.22) can be written as

$$\vartheta (\partial_\mu \psi - \vartheta \mu^{-2} \eta) \partial_t (\vartheta^{-1} \mu) \leq 0, \quad (2.25)$$

and (2.23) follows, provided the time rate of $(\vartheta^{-1} \mu)$ can be chosen arbitrarily.

Next, we specify the free energy density (2.19)₃ as follows (cf. (2.14)):

$$\psi = \widehat{\psi}(\rho, \nabla \rho, \mu, \vartheta) = -\mu \rho + \varphi(\rho, \vartheta) + \frac{1}{2} |\nabla \rho|^2, \quad (2.26)$$

with

$$\varphi(\rho, \vartheta) = f(\rho) - c_v \vartheta (\ln \vartheta - 1) - c_0 \rho (\vartheta - \vartheta_c), \quad c_0 > 0, \quad (2.27)$$

where the double-well potential and the purely caloric free energy are supplemented by a coupling term that is effective when and where the temperature differs from the characteristic temperature ϑ_c . With this final constitutive choice, the consistency condition (2.24) reduces to:

$$\mu \rho + c_0 \vartheta \rho + c_v \vartheta \ln \vartheta = 0; \quad (2.28)$$

moreover, the balance of microforces (2.6) becomes:

$$\kappa \partial_t \rho - \Delta \rho + f'(\rho) - c_0 \vartheta = \mu \quad (2.29)$$

¹Note that the last two variables in the list give way to incorporate in the model the dissipation mechanisms relative to, respectively, atom-rearrangement without diffusion and heat conduction. Indeed, it is clear that (2.19) covers both special cases when either temperature or chemical potential is a space-time constant.

where, with slight abuse of notation, we have written $f'(\rho)$ for $(f'(\rho) + c_0\vartheta_c)$. As to the microenergy balance (2.10), we find:

$$\partial_t(-\vartheta^{-1}\mu^2\rho) = \vartheta^{-1}\mu(\bar{\sigma} + \kappa(\partial_t\rho)^2), \quad (2.30)$$

an equation to be compared with (2.2). Our current mathematical model regards processes of phase segregation by atomic re-arrangement in the presence of thermal effects as solutions of the system of equations (2.28), (2.29) and (2.30).

3 Mathematical formulation and results

The A-C system we derived is more difficult to deal with than its temperature-independent version we tackled successfully in [3], let alone the standard A-C equation (2.9). In fact, in addition to a PDE and an ODE as in [3], we now have to take care also of the transcendental equation (2.28). Our strategy is to repeat, for as much as is possible, the procedure in [3]: accordingly, we discuss the ODE (2.30) first together with the relative initial condition, then we pass to the PDE and the transcendental equation, together with the relative boundary and initial conditions.

3.1 Preliminaries

In order to carry out the first part of our program, we adopt a change of variable to give (2.30) plus (2.5) the form of a parametric initial-value problem. We begin by introducing the initial value ϑ_0 of ϑ and setting:

$$-\eta = \vartheta^{-1}\xi = \vartheta^{-1}\mu^2\rho, \quad \xi_0 = -\vartheta_0\eta_0, \quad \eta_0 = -\vartheta_0^{-1}\mu_0^2\rho_0.$$

We then have that

$$\mu = \sqrt{\xi/\rho}, \quad (3.1)$$

whence the Cauchy problem:

$$\vartheta\partial_t(\vartheta^{-1}\xi) + \frac{\kappa(\partial_t\rho)^2 + \bar{\sigma}}{\sqrt{\rho}}\sqrt{\xi} = 0, \quad (\vartheta^{-1}\xi)|_{t=0} = -\eta_0. \quad (3.2)$$

Next, we restrict attention to the class of processes such that

$$\vartheta\partial_t(\vartheta^{-1}\xi) \simeq \partial_t\xi,$$

and replace (3.2) by the simpler problem:

$$\partial_t\xi + \frac{\kappa(\partial_t\rho)^2 + \bar{\sigma}}{\sqrt{\rho}}\sqrt{\xi} = 0, \quad \xi|_{t=0} = \xi_0, \quad (3.3)$$

parameterized on both the space variable x and the field $\rho(x, \cdot)$. Although simpler, this Cauchy problem is by no means trivial, because it can exhibit the Peano phenomenon and have infinitely many solutions; just as we did in [3], we pick a suitably defined *maximal solution* ξ (or $\sqrt{\xi}$), having the important property to stay positive as long as is possible.

Remark 3.1. It remains to be seen whether the class we restrict attention to does include interesting phase-segregation processes. At this time, we cannot do any better than planning to check if this is the case by running numerical simulations.

To complete our program, we note that, with (3.1), (2.29) and (2.28) become, respectively,

$$\kappa \partial_t \rho - \Delta \rho + f'(\rho) - c_0 \vartheta = \sqrt{\xi/\rho}, \quad (3.4)$$

and

$$\lambda(\rho, \vartheta) := c_0 \rho \vartheta + c_v \vartheta \ln \vartheta = -\sqrt{\rho \xi}, \quad (3.5)$$

an *integro-differential system* for ρ and ϑ , where $\sqrt{\xi}$ is implicitly defined in terms of ρ as the maximal solution to (3.3). This system is to be supplemented with the boundary condition (2.3), the initial condition for ρ in (2.4), and a compatible initial condition for ϑ .

3.2 Results

In view of the above discussion, we look for suitably smooth triplets of time-dependent fields (ρ, ξ, ϑ) over a regular region Ω with boundary Γ , such that:

$$0 < \rho < 1, \quad \xi \geq 0, \quad \text{and} \quad \vartheta > 0; \quad (3.6)$$

$$\partial_t \rho - \Delta \rho + f'(\rho) - c_0 \vartheta = \sqrt{\xi/\rho}, \quad \text{with} \quad \partial_n \rho = 0 \quad \text{on} \quad \Gamma; \quad (3.7)$$

$$\partial_t \xi + \frac{|\partial_t \rho|^2 + \bar{\sigma}}{\sqrt{\rho}} \sqrt{\xi} = 0; \quad (3.8)$$

$$\lambda(\rho, \vartheta) = -\sqrt{\rho \xi}; \quad (3.9)$$

$$\rho(0) = \rho_0, \quad \xi(0) = \xi_0, \quad \text{and} \quad \vartheta(0) = \vartheta_0; \quad (3.10)$$

and that, moreover,

$$\xi \text{ is maximal among the } \xi\text{'s satisfying (3.8) and the second of (3.10).} \quad (3.11)$$

The problem's structure is the same as in [3], apart for the modifications due to the presence of the temperature variable ϑ . Two items deserve a supplement of discussion. The first is that, just as in [3], we assume that

$$0 \leq f = f_1 + f_2, \quad \text{where} \quad f_1, f_2 : (0, 1) \rightarrow \mathbb{R} \quad \text{are } C^2\text{-functions,} \quad (3.12)$$

$$f_1 \text{ is convex,} \quad f'_2 \text{ is bounded,} \quad \lim_{r \searrow 0} f'(r) = -\infty, \quad \text{and} \quad \lim_{r \nearrow 1} f'(r) = +\infty, \quad (3.13)$$

with the constant $c_0 \vartheta_c$ thought of as incorporated in $f'_2(\rho)$. The second item has to do with the admissible choices of initial data: not only they must agree with (3.6), and hence satisfy

$$0 < \rho_0 < 1, \quad \xi_0 \geq 0, \quad \vartheta_0 > 0, \quad (3.14)$$

but also with (3.9), that is to say, they have to satisfy

$$\lambda(\rho_0, \vartheta_0) = -\sqrt{\rho_0 \xi_0}. \quad (3.15)$$

To see what restrictions this last condition implies on the choice of ϑ_0 , it is convenient to study the function the function $\lambda_r : s \mapsto \lambda(r, s) = c_0 r s + c_v s \ln s$ on $(0, +\infty)$ for a given $r \in (0, 1)$. Clearly, (i) λ_r is strictly convex and tends to 0 as s tends to 0; moreover, (ii) the equation $\lambda_r(s) = 0$ has in $(0, +\infty)$ a unique solution, that we denote by $\bar{s}(r)$; finally, (iii) λ_r has a unique minimum point, denoted by $\underline{s}(r)$; in summary, for each fixed $r \in (0, 1)$,

$$0 < \underline{s}(r) < \bar{s}(r), \quad \lambda(r, \bar{s}(r)) = 0, \quad \text{and} \quad \frac{\partial \lambda}{\partial s}(r, \underline{s}(r)) = 0. \quad (3.16)$$

A simple computation shows that

$$\begin{aligned} \underline{s}(r) &= e^{-1-c_* r}, \quad \bar{s}(r) = e^{-c_* r}, \quad \text{and} \\ \lambda(r, \underline{s}(r)) &= -c_v e^{-1-c_* r}, \quad \text{where} \quad c_* := c_0/c_v. \end{aligned} \quad (3.17)$$

Therefore, a necessary condition for the existence of ϑ_0 is that

$$\sqrt{\rho_0 \xi_0} \leq c_v e^{-1-c_* \rho_0} \quad \text{a.e. in } \Omega, \quad (3.18)$$

i.e., that $\sup \zeta \leq 0$, where $\zeta := \sqrt{\rho_0 \xi_0} - c_v e^{-1-c_* \rho_0}$. If such a condition is satisfied, and if we want to solve (a.e. in Ω) the equation $\lambda(\rho_0(x), s) = 0$ for s , then uniqueness holds if $\zeta(x) = 0$, and $\underline{s}(\rho_0(x))$ is the unique solution. Otherwise, if the strict inequality holds, then there are two solutions, the one in the interval $(0, \underline{s}(\rho_0(x)))$ the other in $(\underline{s}(\rho_0(x)), \bar{s}(\rho_0(x)))$.

For existence of a local-in-time solution (ρ, ξ, ϑ) , a modest reinforcement of condition (3.18) and a proper choice of ϑ_0 suffice, namely,

$$\sup(\sqrt{\rho_0 \xi_0} - c_v e^{-1-c_* \rho_0}) < 0 \quad \text{and} \quad \vartheta_0 \geq \underline{s}(\rho_0) \quad \text{a.e. in } \Omega. \quad (3.19)$$

Under these assumptions, we can state the following result.

Theorem 3.2. *Assume that (3.12)–(3.13) and (3.19) hold. Moreover, assume that*

$$\bar{\sigma} \in L^\infty(\Omega \times (0, +\infty)), \quad (\bar{\sigma})^- \in L^\infty(0, \infty; L^1(\Omega)), \quad \text{and} \quad \rho_0, \xi_0, \vartheta_0 \in L^\infty(\Omega); \quad (3.20)$$

$$\rho_0 \in H^3(\Omega), \quad \partial_n \rho_0|_\Gamma = 0, \quad \Delta \rho_0 \in L^\infty(\Omega), \quad \inf \rho_0 > 0, \quad \sup \rho_0 < 1; \quad (3.21)$$

$$\xi_0 \geq 0, \quad \sqrt{\xi_0} \in H^1(\Omega), \quad \lambda(\vartheta_0, \rho_0) = -\sqrt{\rho_0 \xi_0}. \quad (3.22)$$

Then, there exist $T > 0$ and a triplet (ρ, ξ, ϑ) satisfying:

$$\rho \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)); \quad (3.23)$$

$$\rho \in L^p(0, T; W^{2,p}(\Omega)) \text{ for each } p < +\infty, \quad \partial_t \rho \in L^\infty(Q), \quad Q := \Omega \times (0, T); \quad (3.24)$$

$$\xi \in L^\infty(Q) \cap W^{1,1}(0, T; L^1(\Omega)), \quad \vartheta \in L^\infty(Q), \quad \partial_t \vartheta \in L^\infty(Q); \quad (3.25)$$

$$\inf \rho > 0, \quad \sup \rho < 1, \quad \inf \vartheta > 0, \quad (3.26)$$

and solving problem (3.7)–(3.11).

In the next section, this existence result is proved by a fixed-point argument. The method we use seems to be of some interest, because it relies on the application of the Tychonoff-Schauder theorem in a weak topology.

4 Proof of Theorem 3.2

By (3.19), we can start by choosing $\varepsilon_0 > 0$ such that

$$\sqrt{\rho_0 \xi_0} \leq c_v e^{-1-c_* \rho_0} - 2\varepsilon_0 \quad \text{a.e. in } \Omega. \quad (4.1)$$

Moreover, as the result is local, we fix a reference final time $T^* > 0$ (e.g., $T^* = 1$) and assume $T \leq T^*$ in the sequel. Our method is the following. By looking at (3.7)–(3.8) and to (3.9), separately, we construct two maps

$$\mathcal{F}_1 : \vartheta \mapsto (\rho, \xi) \quad \text{and} \quad \mathcal{F}_2 : (\rho, \xi) \mapsto \vartheta$$

with proper domains. Namely, the domain of \mathcal{F}_1 is a convex set \mathcal{K} depending on T and on a further parameter M , and the domain of \mathcal{F}_2 is the range \mathcal{R} of \mathcal{F}_1 . Then, we prove that a suitable choice of T and M ensures that the range of \mathcal{F}_2 is contained in \mathcal{K} . This allows us to look for a fixed point of $\mathcal{F}_2 \circ \mathcal{F}_1$. To this aim, we want to use the Tychonoff-Schauder theorem. For that reason, \mathcal{K} will be endowed with some weak topology. We start to construct \mathcal{F}_1 . The whole argument relies on the technique of [3] and the whole paper has to be revisited. This is done in the next steps. In particular, as in [3], we have to consider both the Cauchy problem obtained by coupling equation (3.8) to the second (3.10) and the Cauchy problem

$$\partial_t \xi + \frac{|\partial_t v|^2 + \bar{\sigma}}{\sqrt{v}} \sqrt{\xi} = 0 \quad \text{and} \quad \xi(0) = \xi_0 \quad (4.2)$$

(i.e., ρ is replaced by v in (3.8)), where v satisfies

$$v \in D(\Phi) := \{v \in H^1(0, T; L^2(\Omega)) : v > 0, 1/v \in L^\infty(Q)\} \quad (4.3)$$

and possibly further conditions later on. Moreover, the map $\Phi : D(\Phi) \rightarrow L^\infty(0, T; L^1(\Omega))$ is defined by $\Phi(v) = \sqrt{\xi}$, where ξ is the maximal solution to the Cauchy problem (4.2).

Remark 4.1. In the sequel, our notation is going to reflect dependences, if any, on such parameters as, say, T and M ; however, possible dependences on problem data such as Ω , f , $\bar{\sigma}$, and the initial data, will not be displayed.

The crucial constants and the maximum principle. We set:

$$\vartheta_* := \inf_{0 < r < 1} \underline{s}(r) \quad \text{and} \quad \vartheta^* := \sup_{0 < r < 1} \bar{s}(r). \quad (4.4)$$

A simple calculation yields:

$$0 < \vartheta_* = e^{-(1+c_*)} \quad \text{and} \quad \vartheta^* = 1;$$

thus, we require that ϑ obeys the following double limitation:

$$\vartheta_* \leq \vartheta \leq \vartheta^*; \quad (4.5)$$

in particular, the condition $\inf \vartheta > 0$ (see (3.26)) will automatically hold true. Now, we notice that (3.7) differs from the analogous one of [3] just for the presence of $-c_0\vartheta$ on the left-hand side. Therefore, even though such a term is space and time dependent, for a fixed ϑ , it can be seen as a part of the smooth perturbation f_2 of the nonlinear term. Just by thinking of that, the construction of the crucial constants ρ_* , ρ^* , and ξ^* can be done exactly as in the quoted paper, provided that the definition of M_2 is modified as follows

$$M_2 := \sup\{|f_2'(r) - c_0s| : \rho \in (0, 1), s \in (\vartheta_*, \vartheta^*)\}. \quad (4.6)$$

Then, the analogues of Lemmas 4.1-4.3 hold in the present case, provided that we assume (4.5). Indeed, it suffices to read $f_2'(\rho) - c_0\vartheta$ in place of $f_2'(\rho)$ in the proofs.

The convex set. We define the set \mathcal{K} as follows:

$$\mathcal{K} = \{\vartheta \in L^2(Q) : \vartheta, \partial_t \vartheta \in L^\infty(Q), \vartheta_* \leq \vartheta \leq \vartheta^*, |\partial_t \vartheta| \leq M\}. \quad (4.7)$$

Clearly, \mathcal{K} depends on T and on the a real parameter M , even though such a dependence is not stressed in the notation. Moreover, it is non-empty, convex, bounded, and closed.

In the next steps, $\vartheta \in \mathcal{K}$ is given and we want to solve (3.7)–(3.8) and the first two equations (3.10) for (ρ, ξ) by applying the procedures of [3].

L^p estimates. We make a general observation. If $p \in (1, +\infty)$ and some function z solves

$$\begin{aligned} \partial_t z - \Delta z &= g \in L^p(Q^*) \quad \text{in } Q^* := \Omega \times (0, T^*) \quad \text{and} \quad \partial_n z = 0 \text{ on the boundary,} \\ z(0) &= z_0 \quad \text{with} \quad z_0 \in L^\infty(\Omega), \quad \Delta z_0 \in L^\infty(\Omega), \quad \text{and} \quad \partial_n z_0 = 0 \text{ on the boundary,} \end{aligned}$$

then, the following estimate holds

$$\|\partial_t z\|_{L^p(Q^*)} + \|z\|_{L^p(0, T^*; W^{2,p}(\Omega))} \leq C_p \left(\|g\|_{L^p(Q^*)} + \|z_0\|_{L^\infty(\Omega)} + \|\Delta z_0\|_{L^\infty(\Omega)} \right),$$

where C_p depends on Ω , T^* , and p , only. Therefore, the same constant C_p yields

$$\|\partial_t z\|_{L^p(Q)} + \|z\|_{L^p(0, T; W^{2,p}(\Omega))} \leq C_p \left(\|g\|_{L^p(Q)} + \|z_0\|_{L^\infty(\Omega)} + \|\Delta z_0\|_{L^\infty(\Omega)} \right) \quad (4.8)$$

for the solution z to the problem

$$\partial_t z - \Delta z = g \in L^p(Q) \quad \text{in } Q, \quad \partial_n z = 0 \text{ on the boundary} \quad \text{and} \quad z(0) = z_0, \quad (4.9)$$

provided that $T \leq T^*$. Now, note that $z := \rho$ solves (4.9) with $g := -f'(\rho) + c_0\vartheta + \sqrt{\xi/\rho}$ and that we are assuming $\vartheta \in \mathcal{K}$; moreover, (3.20)–(3.21) hold. Thus, by applying (4.8), we have:

$$\|\partial_t \rho\|_{L^p(Q)} + \|\rho\|_{L^p(0, T; W^{2,p}(\Omega))} \leq R_p, \quad (4.10)$$

where R_p depends on the same parameters as in [3] and on ϑ^* , but not on M or T (following the rule laid down in Remark 4.1, our notation stresses just the dependence on p). Furthermore, all this is true for the solution ρ coming from $\sqrt{\xi} = \Phi(v)$, where $v \in D(\Phi)$ satisfies $\rho_* \leq v \leq \rho^*$.

L^∞ estimate. As in [3], we differentiate with respect to time and see that $u := \partial_t \rho$ satisfies

$$\partial_t u - \Delta u + u = F + c_0 \partial_t \vartheta \quad \text{and} \quad \partial_n u = 0 \text{ on the boundary}, \quad (4.11)$$

$$u(0) = \Delta \rho_0 - f'(\rho_0) + \sqrt{\xi_0/\rho_0} + c_0 \vartheta_0, \quad (4.12)$$

where F is as in [3], namely,

$$F := \partial_t \rho - f''(\rho) \partial_t \rho - \frac{1}{2} \varphi \rho^{-3/2} \partial_t \rho - \frac{1}{2} \chi (|\partial_t v|^2 + \bar{\sigma}) (v \rho)^{-1/2}, \quad (4.13)$$

where $\varphi = \Phi(v)$, with $v \in D(\Phi)$ such that $\rho_* \leq v \leq \rho^*$, and χ is some characteristic function. Then, we fix the right value of q as in [3], in order to get the desired estimate. We have

$$\|u\|_{L^\infty(Q)} \leq C \left(\|F + c_0 \partial_t \vartheta\|_{L^q(Q)} + \|u(0)\|_{L^\infty(\Omega)} \right)$$

where C does not depend on T (by the above general observation). As $\vartheta \in \mathcal{K}$, we have:

$$\|F + c_0 \partial_t \vartheta\|_{L^q(Q)} \leq \|F\|_{L^q(Q)} + c_0 (|\Omega|T)^{1/q} M;$$

by the use of the L^p estimates, we obtain:

$$\|\partial_t \rho\|_{L^\infty(Q)} \leq R_\infty + C_\infty M T^{1/q}, \quad (4.14)$$

where R_∞ , C_∞ , and q are independent of M and T .

The first map. At this point, for every $\vartheta \in \mathcal{K}$, we consider the following problem: in the set

$$\begin{aligned} \tilde{\mathcal{R}} = \Big\{ (\rho, \xi) : & \rho \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)), \quad \xi \in W^{1,1}(0, T; L^1(\Omega)) \\ & \rho_* \leq \rho \leq \rho^*, \quad 0 \leq \xi \leq \xi^* \\ & \|\partial_t \rho\|_{L^p(Q)} + \|\rho\|_{L^p(0, T; W^{2,p}(\Omega))} \leq R_p \text{ for every } p \in (1, +\infty) \\ & \|\partial_t \rho\|_{L^\infty(Q)} \leq R_\infty + C_\infty M T^{1/q} \Big\}, \end{aligned} \quad (4.15)$$

find a pair

$$(\rho, \xi) \text{ satisfying (3.7), (3.8), the first two of (3.10), and (3.11)} \quad (4.16)$$

Indeed, the analogue of the map Ψ considered in [3] can be defined in the same way and actually maps its domain into itself also in the present case, because of our choice of the constants. Moreover, in performing the contraction estimate, just differences have to be considered and ϑ disappears. Therefore, by setting:

$$\text{for } \vartheta \in \mathcal{K}, \quad \mathcal{F}_1(\vartheta) \text{ is the solution } (\rho, \xi) \in \tilde{\mathcal{R}} \text{ to problem (4.16),} \quad (4.17)$$

we obtain a well-defined map $\mathcal{F}_1 : \mathcal{K} \rightarrow \tilde{\mathcal{R}}$. We set:

$$\mathcal{R} \text{ is the range of } \mathcal{F}_1. \quad (4.18)$$

Such a subset of $\tilde{\mathcal{R}}$ depends on T and M and our next step is the definition of a map $\mathcal{F}_2 : \mathcal{R} \rightarrow \mathcal{K}$ that works as follows: given $(\rho, \xi) \in \mathcal{R}$, the value $\mathcal{F}_2(\rho, \xi)$ is a function ϑ satisfying (3.9) and the third of (3.10). However, before doing that, some more work is needed, since we are not sure that such a map is well-defined. It is not, indeed, unless T and M satisfy suitable constraints, and the next steps are devoted to find them.

Higher regularity estimates. Let $(\rho, \xi) \in \mathcal{R}$. Then, (ρ, ξ) solves (4.16) for some $\vartheta \in \mathcal{K}$, whence $u := \partial_t \rho$ solves (4.11)–(4.12) where F is the same as (4.13). On noting that (3.21)–(3.22) imply $u(0) \in H^1(\Omega)$, we have by the above L^p estimates and the general theory that

$$\begin{aligned} \|\partial_t^2 \rho\|_{L^2(Q)} + \|\partial_t \rho\|_{L^2(0,T;H^1(\Omega))} &= \|\partial_t u\|_{L^2(Q)} + \|u\|_{L^2(0,T;H^1(\Omega))} \\ &\leq c(\|F + c_0 \partial_t \vartheta\|_{L^2(Q)} + \|u(0)\|_{H^1(\Omega)}) \leq C'(T, M), \end{aligned} \quad (4.19)$$

where the form of the dependence of $C'(T, M)$ on T and M is not important in the sequel.

New a priori estimate. Let $(\rho, \xi) \in \mathcal{R}$. Then, (ρ, ξ) solves (4.16) for some $\vartheta \in \mathcal{K}$. Therefore, as in [3], we have

$$\partial_t \sqrt{\xi} = -\chi \frac{|\partial_t \rho|^2 + \bar{\sigma}}{2\sqrt{\rho}},$$

where χ is some characteristic function, whence immediately

$$\|\partial_t \sqrt{\xi}\|_{L^\infty(Q)} \leq \frac{1}{2\sqrt{\rho_*}} (\|\partial_t \rho\|_{L^\infty(Q)}^2 + \|\bar{\sigma}\|_{L^\infty(\Omega \times (0, +\infty))}).$$

Hence, if c stands for different constants independent of M and T , we deduce:

$$\begin{aligned} \left| \partial_t \sqrt{\rho \xi} \right| &= \left| \frac{\partial_t \rho}{2\sqrt{\rho}} \sqrt{\xi} + \sqrt{\rho} \partial_t \sqrt{\xi} \right| \leq \frac{\sqrt{\xi^*}}{2\sqrt{\rho_*}} \|\partial_t \rho\|_{L^\infty(Q)} + \sqrt{\rho^*} \|\partial_t \sqrt{\xi}\|_{L^\infty(Q)} \\ &\leq c \left(1 + \|\partial_t \rho\|_{L^\infty(Q)}^2 \right) \leq c \left(1 + (R_\infty + C_\infty M T^{1/q})^2 \right), \end{aligned}$$

and we conclude that

$$\left| \partial_t \sqrt{\rho \xi} \right| \leq C_1 (1 + M^2 T^{2/q}), \quad (4.20)$$

where C_1 is independent of T and M .

First restriction on parameters. Observe that

$$\left| \partial_t (\sqrt{\rho \xi} - c_v e^{-1-c_* \rho}) \right| \leq |\partial_t \sqrt{\rho \xi}| + c_v c_* |\partial_t \rho|.$$

By accounting for (4.20) and (4.14), we deduce that

$$\|\partial_t (\sqrt{\rho \xi} - c_v e^{-1-c_* \rho})\|_{L^\infty(Q)} \leq \frac{C_2}{2} (1 + M^2 T^{2/q} + M T^{1/q}) \leq C_2 (1 + M^2 T^{2/q}),$$

where C_2 is independent of T and M ; this implies that

$$\|(\sqrt{\rho \xi} - c_v e^{-1-c_* \rho}) - (\sqrt{\rho_0 \xi_0} - c_v e^{-1-c_* \rho_0})\|_{L^\infty(Q)} \leq C_2 T (1 + M^2 T^{2/q}).$$

By (4.1), we conclude that

$$\sqrt{\rho\xi} \leq c_v e^{-1-c_*\rho} - \varepsilon_0 \quad \text{a.e. in } Q, \quad (4.21)$$

whenever M and T satisfy

$$C_2 T(1 + M^2 T^{2/q}) \leq \varepsilon_0. \quad (4.22)$$

Therefore, if T and M satisfy (4.22), every $(\rho, \xi) \in \mathcal{R}$ fulfils (4.21).

The second map. Assume $(\rho, \vartheta) \in \mathcal{R}$ and (4.22). By (4.21), we have, in particular, that

$$\sqrt{\rho\xi} < c_v e^{-1-c_*\rho} \quad \text{a.e. in } Q.$$

Therefore, for a.a. $(x, t) \in Q$, the equation $\lambda(\rho(x, t), s) = -\sqrt{\rho(x, t)\xi(x, t)}$ has two solutions s_1 and s_2 which belong to $(0, \underline{s}(\rho(x, t)))$ and $(\underline{s}(\rho(x, t)), \bar{s}(\rho(x, t)))$, respectively. We term the latter $\vartheta_2(x, t)$ for a while and obtain a bounded function, thus a function $\vartheta_2 \in L^2(Q)$. Then, we can define $\mathcal{F}_2 : \mathcal{R} \rightarrow L^2(Q)$ by setting: $\mathcal{F}_2(\rho, \xi)$ is such a function ϑ_2 . Therefore, for $(\rho, \xi) \in \mathcal{R}$,

$$\vartheta = \mathcal{F}_2(\rho, \xi) \quad \text{means} \quad \underline{s}(\rho) < \vartheta < \bar{s}(\rho) \quad \text{and} \quad \lambda(\rho, \vartheta) = -\sqrt{\rho\xi} \quad \text{a.e. in } Q. \quad (4.23)$$

As both ρ and ξ are continuous with respect to time (for a.a. $x \in \Omega$) and the function $\vartheta := \mathcal{F}_2(\rho, \xi)$ is always different from $\underline{s}(\rho)$, time continuity holds for ϑ as well, and it is clear that $\vartheta(0) = \vartheta_0$.

Although \mathcal{F}_2 is well-defined, we still have to find the restriction on T and M that ensures that the range of \mathcal{F}_2 is contained in \mathcal{K} .

Estimate from below. Let $(\rho, \xi) \in \mathcal{R}$, set $\vartheta := \mathcal{F}_2(\rho, \xi)$, and assume (4.22). Then (4.21) holds. For almost every $(x, t) \in Q$, we write the second-order Taylor expansion of the function $s \mapsto \lambda(\rho(x, t), s)$ with center at $s = \underline{s}(\rho(x, t))$ (hereafter, to lighten our notation, we refrain from writing (x, t)). We find $s' \in (\underline{s}(\rho), \vartheta)$ such that the following holds:

$$\begin{aligned} -\sqrt{\rho\xi} &= \lambda(\rho, \vartheta) = \lambda(\rho, \underline{s}(\rho)) + \frac{\partial \lambda}{\partial s}(\rho, \underline{s}(\rho)) (\vartheta - \underline{s}(\rho)) + \frac{1}{2} \frac{\partial^2 \lambda}{\partial s^2}(\rho, s') (\vartheta - \underline{s}(\rho))^2 \\ &= -c_v e^{-1-c_*\rho} + \frac{c_v}{2s'} (\vartheta - \underline{s}(\rho))^2, \end{aligned}$$

in view of (3.16) and (3.17). By (4.22), we deduce that

$$\frac{c_v}{2s'} (\vartheta - \underline{s}(\rho))^2 = c_v e^{-1-c_*\rho} - \sqrt{\rho\xi} \geq \varepsilon_0,$$

whence

$$(\vartheta - \underline{s}(\rho))^2 \geq \frac{2\varepsilon_0}{c_v} s' \geq \frac{2\varepsilon_0}{c_v} \underline{s}(\rho) = \frac{2\varepsilon_0}{c_v} e^{-1-c_*\rho} \geq \frac{2\varepsilon_0}{c_v} e^{-1-c_*\rho^*}.$$

As $\vartheta > \underline{s}(\rho)$, we conclude that

$$\vartheta - \underline{s}(\rho) \geq 2\delta_0 \quad \text{a.e. in } Q \quad (4.24)$$

(with an obvious definition of δ_0 ; note that δ_0 is independent of T and M).

Lemma 4.2. Assume that $r \in (\rho_*, \rho^*)$, $s \in (\vartheta_*, \vartheta^*)$, and $\delta > 0$. Then,

$$s \geq \underline{s}(r) + \delta \quad \text{implies} \quad \frac{\partial \lambda}{\partial s}(r, s) \geq c_v \ln(1 + \delta/\vartheta^*). \quad (4.25)$$

Proof. Indeed, if $s \geq \underline{s}(r) + \delta$, we have that

$$\begin{aligned} \frac{\partial \lambda}{\partial s}(r, s) &= c_0 r + c_v + c_v \ln s = c_0 r + c_v + c_v \ln \underline{s}(r) + c_v \ln \frac{s}{\underline{s}(r)} \\ &= \frac{\partial \lambda}{\partial s}(r, \underline{s}(r)) + c_v \ln \frac{s}{\underline{s}(r)} = c_v \ln \frac{s}{\underline{s}(r)} \geq c_v \ln \frac{\underline{s}(r) + \delta}{\underline{s}(r)} = c_v \ln(1 + \delta/\underline{s}(r)) \end{aligned}$$

and the desired conclusion follows, because $\underline{s}(r) \leq \vartheta^*$. \square

Estimate of the time derivative. Assume that $(\rho, \xi) \in \mathcal{R}$, $\vartheta := \mathcal{F}_2(\rho, \xi)$, and that (4.22) holds. On the one hand, we have that

$$\frac{\partial \lambda}{\partial s}(\rho, \vartheta) \partial_t \vartheta = -\partial_t \sqrt{\rho \xi} - \frac{\partial \lambda}{\partial r}(\rho, \vartheta) \partial_t \rho = -\partial_t \sqrt{\rho \xi} - c_0 \vartheta \partial_t \rho \quad \text{a.e. in } Q.$$

On the other hand, (4.24) holds. Therefore, if we apply the previous lemma for $\delta := 2\delta_0$ and set $L_0 := (c_v \ln(1 + 2\delta_0/\vartheta^*))^{-1}$, we obtain:

$$|\partial_t \vartheta| \leq L_0 (|\partial_t \sqrt{\rho \xi}| + c_0 \vartheta |\partial_t \rho|) \leq L_0 (|\partial_t \sqrt{\rho \xi}| + c_0 \vartheta^* |\partial_t \rho|). \quad (4.26)$$

Now, we account for (4.20) and conclude that

$$|\partial_t \vartheta| \leq L_0 C_1 (1 + M^2 T^{2/q}) + L_0 c_0 \vartheta^* (R_\infty + C_\infty M T^{1/q}) \quad \text{a.e. in } Q. \quad (4.27)$$

At this point, we are ready to choose the constants T and M .

Choice of the constants. Clearly, (4.27) implies $|\partial_t \vartheta| \leq M$, whenever

$$L_0 C_1 (1 + M^2 T^{2/q}) + L_0 c_0 \vartheta^* (R_\infty + C_\infty M T^{1/q}) \leq M. \quad (4.28)$$

Therefore, we choose, e.g., $M = L_0 C_1 + L_0 c_0 \vartheta^* R_\infty + 1$ and T small enough for both (4.22) and (4.28) to hold. With such a choice, the range of \mathcal{F}_1 is contained in \mathcal{R} and \mathcal{F}_2 is a well-defined map whose range is contained in \mathcal{K} . Hence, $\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1$ is a well-defined map from \mathcal{K} into $L^2(Q)$ that satisfies $\mathcal{F}(\mathcal{K}) \subseteq \mathcal{K}$.

Conclusion of the proof. As anticipated, we aim to an application of the Tychonoff-Schauder fixed point theorem. As far as the topology of \mathcal{K} is concerned, we see \mathcal{K} as a subset of the topological vector space obtained by endowing $L^2(Q)$ with the weak topology. Therefore, the convex set \mathcal{K} is compact. So, the last point of our proof is the continuity of \mathcal{F} with respect to the topology of \mathcal{K} . To this aim, we observe that $L^2(Q)$ with its strong topology is reflexive and separable and that \mathcal{K} is bounded. Thus, the topology of \mathcal{K} comes from a metric. In particular, \mathcal{F} is continuous if and only if it is sequentially continuous. So, we pick $\bar{\vartheta}_n, \bar{\vartheta}$ such that

$$\bar{\vartheta}_n \in \mathcal{K} \quad \text{for every } n \quad \text{and} \quad \bar{\vartheta}_n \rightarrow \bar{\vartheta} \quad \text{weakly in } L^2(Q) \quad (4.29)$$

and pass to prove that the corresponding functions $\vartheta_n := \mathcal{F}(\bar{\vartheta}_n)$ and $\vartheta := \mathcal{F}(\bar{\vartheta})$ satisfy

$$\vartheta_n \rightarrow \vartheta \quad \text{weakly in } L^2(Q). \quad (4.30)$$

It suffices for us to show that every subsequence of $\{\vartheta_n\}$ has a subsequence that converges to ϑ weakly in $L^2(Q)$. In the sequel, to lighten our notation, we denote by the same symbol ($\{y_n\}$, say) both a sequence and all its subsequences. So, we start by any subsequence $\{\vartheta_n\}$ of the given sequence and look for a subsequence of it satisfying (4.30). We set for convenience $(\rho_n, \xi_n) := \mathcal{F}_1(\bar{\vartheta}_n)$. Therefore, we have $(\rho_n, \xi_n) \in \mathcal{R} \subset \widetilde{\mathcal{R}}$ and all the estimates given in definition (4.15) hold true, as well as (4.19). In particular, we have

$$\rho_n \rightarrow \rho \quad \text{weakly in } W^{1,p}(0, T; L^p(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega)) \text{ for every } p < +\infty \quad (4.31)$$

$$\partial_t \rho_n \rightarrow \partial_t \rho \quad \text{weakly in } H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad (4.32)$$

for some ρ , at least for a subsequence. From (4.31)–(4.32) we also have that

$$\rho_n \rightarrow \rho \quad \text{uniformly in } Q \quad \text{and} \quad \partial_t \rho_n \rightarrow \partial_t \rho \quad \text{strongly in } L^1(Q). \quad (4.33)$$

As $\rho_* \leq \rho_n \leq \rho^*$ for every n , we infer the uniform convergence of $f'(\rho_n)$ to $f'(\rho)$. Now, we consider the maximal solution ξ to the Cauchy problem obtained by coupling (3.8) (with such a ρ) and the second (3.10). We prove that $(\rho, \xi) = \mathcal{F}_1(\bar{\vartheta})$. By using [3, Lemma 4.8], we easily deduce that, a.e. in Ω and for every n and $t \in [0, T]$,

$$\begin{aligned} |\sqrt{\xi_n(t)} - \sqrt{\xi(t)}| &\leq c \int_0^T \left| |\partial_t \rho_n(s)|^2 - |\partial_t \rho(s)|^2 \right| ds + c \int_0^T |\rho_n(s) - \rho(s)| ds \\ &\leq c \int_0^T |\partial_t \rho_n(s) - \partial_t \rho(s)| ds + c \int_0^T |\rho_n(s) - \rho(s)| ds, \end{aligned}$$

where c stands for different constants independent of n . We infer that

$$\|\sqrt{\xi_n} - \sqrt{\xi}\|_{L^\infty(0, T; L^1(\Omega))} \leq c(\|\partial_t \rho_n - \partial_t \rho\|_{L^1(Q)} + \|\rho_n - \rho\|_{L^1(Q)}),$$

and, owing to (4.33), we deduce that $\sqrt{\xi_n}$ converges to $\sqrt{\xi}$ in $L^\infty(0, T; L^1(\Omega))$, thus a.e. in Q at least for a subsequence. At this point, it is clear that (ρ, ξ) satisfies problem (3.7)–(3.11) where we read $\bar{\vartheta}$ in place of ϑ . Indeed, (3.8), the second (3.10), and (3.11) hold by definition of ξ . On the other hand, it is straightforward to let n tend to infinity in the variational equation, or in an integrated version of it, and in the Cauchy conditions. We conclude that $(\rho, \xi) = \mathcal{F}_1(\bar{\vartheta})$, and the next step is the convergence of $\{\vartheta_n\}$ to ϑ weakly in $L^2(Q)$, at least for a subsequence. As a matter of fact, we can prove a strong-convergence result. Recall that the estimate from below (4.24) holds for ϑ_n . It follows that

$$\vartheta_n \geq \underline{s}(\rho_n) + 2\delta_0 = e^{-1-c_*\rho_n} + 2\delta_0 = e^{-1-c_*\rho_n} - e^{-1-c_*\rho} + \underline{s}(\rho) + 2\delta_0,$$

whence

$$\vartheta_n \geq \underline{s}(\rho) + \delta_0 \quad \text{provided that} \quad |\rho_n - \rho| \leq \eta_0,$$

where $\eta_0 > 0$ is given by the uniform continuity of the exponential on every bounded interval (namely, $r, r' \in [\rho_*, \rho^*]$ and $|r - r'| \leq \eta_0$ imply that $|e^{-1-c_*r} - e^{-1-c_*r'}| \leq \delta_0$). Moreover, the similar inequality

$$s \geq \underline{s}(r) + \delta_0$$

holds true whenever $|\rho_n - \rho| \leq \eta_0$ and

$$\min\{\rho_n, \rho\} \leq r \leq \max\{\rho_n, \rho\} \quad \text{and} \quad \min\{\vartheta_n, \vartheta\} \leq s \leq \max\{\vartheta_n, \vartheta\}. \quad (4.34)$$

On the other hand, the uniform convergence (4.33) implies that $|\rho_n - \rho| \leq \eta_0$ in Q for n large enough. Therefore, for such values of n , the following holds. We apply the Lagrange mean value theorem to λ . For a.a. $(x, t) \in Q$ (once again, we omit writing (x, t) in the sequel) and suitable (space and time dependent) r_n and s_n satisfying (4.34), we have that

$$\sqrt{\rho_n \xi_n} - \sqrt{\rho \xi} = \lambda(\rho, \vartheta) - \lambda(\rho_n, \vartheta_n) = \frac{\partial \lambda}{\partial r}(r_n, s_n) (\rho - \rho_n) + \frac{\partial \lambda}{\partial s}(r_n, s_n) (\vartheta - \vartheta_n).$$

Now, we observe that $r_n \in [\rho_*, \rho^*]$ and $s_n \geq \underline{s}(r_n) + \delta_0$ since $|\rho - r_n| \leq \eta_0$. Hence, we can apply Lemma 4.2 and arrive at (4.25) with $r = r_n$, $s = s_n$, and $\delta = \delta_0$. This implies that

$$\ln(1 + \delta_0/\vartheta^*) |\vartheta_n - \vartheta| \leq |\sqrt{\rho_n \xi_n} - \sqrt{\rho \xi}| + c_0 |\rho_n - \rho|.$$

As the right-hand side tends to 0 a.e. in Q and is bounded in $L^\infty(Q)$, we deduce that ϑ_n strongly converges to ϑ in $L^2(Q)$. This concludes the proof.

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